Verification of the $(1-\delta)$-Correctness Proof of Kyber

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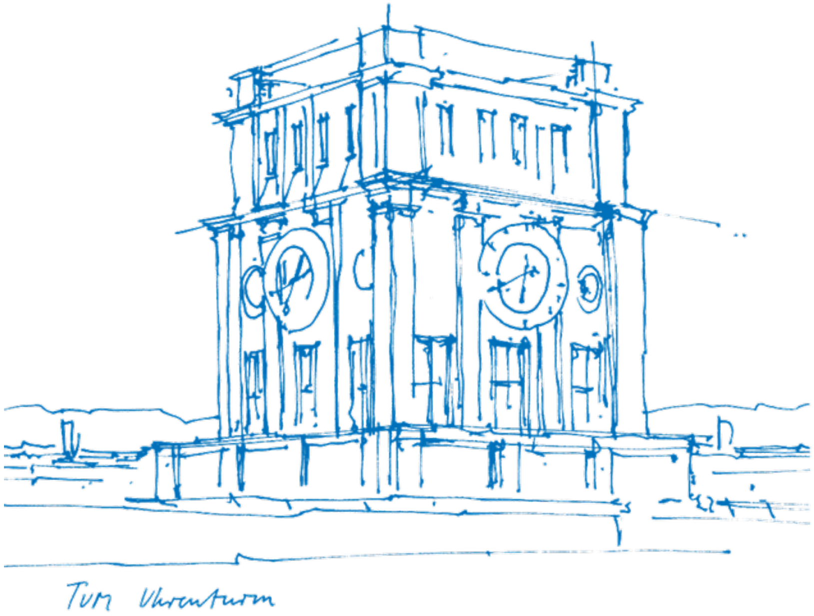

Isabelle

Isabelle is an interactive theorem prover used to formalize and verify mathematics and computer science.

A special ring. . .

$$R_q = \mathbb{Z}_q[x]/(x^n + 1)$$

q prime

n a power of 2

A special ring. . .

Using Isabelle theories for

- finite fields
- polynomials
- equivalence relations
- quotient ring structure
- algebra
- analysis
- . . .

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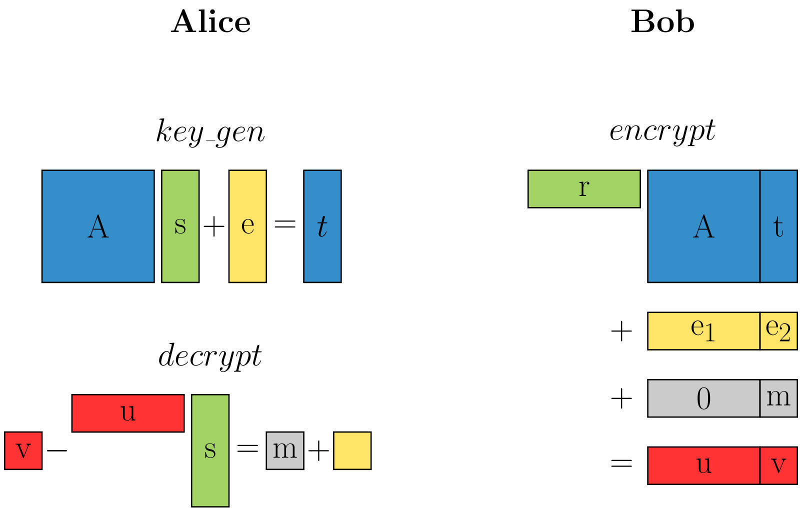
Kyber - a post-quantum crypto system



public key private key errors

message (plaintext) encrypted message (ciphertext)

Kyber Algorithms in Isabelle

```
definition key_gen where
key_gen dt A s e = compress_vec dt (A * s + e)

definition encrypt where
encrypt t A r e1 e2 dt du dv m =
  (compress_vec du (A^T * r + e1),
   compress_poly dv ((decompress_vec dt t)^T * r + e2 +
      to_module (round(q/2)) * bitstring_to_module m))

definition decrypt where
decrypt u v s du dv = compress_poly 1
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Compression and Decompression induce:

- smaller key sizes
- compression errors
- a problem in the $(1-\delta)$-correctness proof
- a problem in the IND-CPA security proof

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A cryptographic scheme is $(1-\delta)$-correct if and only if for all messages m it holds:

$$\mathbb{P}[m = decrypt(sk, encrypt(pk, m)) \mid (sk, pk) \leftarrow key\_gen] \geq 1 - \delta$$

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Correctness of Kyber

Define $\delta := \mathbb{P}[\|e^T r + e_2 - s^T e_1 + e'\|_\infty \geq \lceil q/4 \rfloor]$. Then Kyber is $(1-\delta)$-correct.

Problem in $(1-\delta)$-correctness proof

compression error: $\|x - decompress_d(compress_d(x))\|_\infty$

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$$\hat{x} = x \mod^{\pm} q \quad \text{iff} \quad \hat{x} \in (-\tfrac{q}{2}, \tfrac{q}{2}] \text{ and } \hat{x} \equiv x \mod q$$

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$\Rightarrow \|\cdot\|_\infty$ is not a norm $\lightning$

How to solve this problem?

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$\Rightarrow$ not noticed since chosen parameters fulfil this property

$$7681 \equiv 1 \mod 4 \quad \text{and} \quad 3329 \equiv 1 \mod 4$$

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✓ $(1 - \delta)$-correctness proof is valid

Conclusion

Formalization and verification in Isabelle:

- Kyber PKE algorithms for key generation, encryption and decryption
- $(1-\delta)$-correctness proof of Kyber
- based on "CRYSTALS-Kyber: a CCA-secure module-lattice-based KEM" by Bos et al.

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Problems during formalization:

- $\|\cdot\|_\infty$ not a norm
- Proof was corrected using additional assumption $q \equiv 1 \mod 4$
- $q \equiv 1 \mod 4$ given by NTT properties

Thank you for your attention!

Question: How does the IND-CPA proof for Kyber PKE with compression/decompression work?

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